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Reorientation of a Structure in Space Using a Three-Link Rigid Manipulator

Ranjan Mukherjee* and Mary Zurowski†
Naval Postgraduate School, Monterey, California 93943

The attitude control of space structures is an important problem. There has been significant research in this area that has focussed on the use of momentum exchange devices. In this paper, we propose to reorient space structures that maintain zero angular momentum using a serial link rigid manipulator with three revolute joints. This unconventional method of attitude control exploits the nonholonomic nature of the constraints that arise due to the conservation of angular momentum. We adopt a surface integral approach for the motion planning of the manipulator. The salient features of our approach are (1) it is possible to mathematically prove the reachability of any arbitrary orientation of the space structure, and (2) the motion of the manipulator can be planned amidst additional constraints such as joint limits of the manipulator. Simulation results of a large rotational maneuver are provided.

Nomenclature

frame I	= inertia frame
frame K	= kth link frame of the manipulator according to the Denavit Hartenburg convention, for $k = 1, 2, 3$; $k = 0$ denotes the manipulator base frame
frame S	= frame fixed at the center of mass of the space structure and directed along the principal axes of the structure
$\boldsymbol{I}_k \in R^{3\times 3}$	= inertia matrix of the kth body about the principal axes located at the center of mass and expressed in the kth link frame.
I_{kij}, I_{Sij}	= (i, j)th element of $\boldsymbol{I}_k$, $\boldsymbol{I}_S$, kgm^2
$\boldsymbol{I}_S$	= inertia matrix of the space structure, kgm^2
m_k	= mass of the kth body for $k = 1, 2, 3$
m_S	= mass of the space structure, kg
$\boldsymbol{R}[\bullet, *] \in R^{3\times 3}$	= orthogonal rotation matrix corresponding to a rotation about the $(\bullet)$ axis fixed on the space structure by an angle $(*)$
x_0, y_0, z_0	= position of the center of mass of the space structure in the inertia frame, m
β_0, β_1, β_2, β_3	= Euler parameters
$(\theta_1, \theta_2, \theta_3)$	= joint configuration of the three link manipulator, rad
ϕ_1, ϕ_2, ϕ_3	= y-x-y Euler angles, describing the orientation of the space structure with respect to the inertia frame, rad

I. Introduction

THE attitude control of space structures has been studied over the years by various researchers. Such research has been motivated by the fact that structures in space are required for high precision tasks. The research is very pertinent because the space environment causes a drift in the orientation of free flying bodies. The change could take place due to differential gravitational forces or effects due to solar radiation. In the case of a space station, the change could take place rather rapidly due to dynamic interaction between the space station and robots performing tasks on board the space station. Large changes in the orientation of the space station could take place due to docking with the Shuttle, or due to the operation of booster rockets required for orbit maintenance.

An extensive literature survey of attitude control of space structures can be carried out. It is widely accepted that the best way for attitude correction or stabilization is to use momentum exchange devices, and control momentum gyroscopes are regarded as one of the most desirable devices for attitude control. In this paper we show that the reorientation of a space structure is alternatively possible by using a three link robot manipulator. Manipulators with three or more links are expected to be present onboard spacecrafts or at the site of space stations for automation. If these manipulators can effectively reorient space structures, they will serve as backup devices for attitude control in the event of failure of the conventional attitude control devices.

In manned space explorations, astronauts have been expected to reorient themselves in space during extravehicular activity. In 1972, experimental investigations of an astronaut maneuvering scheme was carried out in Ref. 1. Various limb motions were studied that produced nearly pure rotations about each of the three mutually perpendicular axes fixed at the torso of the astronaut. Although astronauts have tried to learn the process of self-induced rotations, it has been seen that cats perform this kind of maneuver with ease. They always land on their feet when dropped from a height with an arbitrary orientation. The cat performs the maneuver through a cyclic motion in which it first bends its spine forward, then to one side, then to the other side, and finally forward again. In 1969 a dynamical explanation of the falling cat problem was provided in Ref. 2. The falling cat problem[2] and the astronaut maneuvering scheme[1] are simple examples, where a system in the absence of angular momentum, can undergo a change in orientation through internal motion.

In 1991, a bidirectional approach to the motion planning of free-flying space robots was proposed in Ref. 3. It was shown that by utilizing the nonholonomy, the vehicle orientation in addition to the joint variables of the manipulator can be controlled by actuating only the joint variables. This work was motivated by the work in Ref. 4, where cyclic motion of the joint variables were proposed to reorient the space vehicle. In both these works, the controllability of the system was simply assumed. In 1991, Fernandes et al.[5] proved the controllability of a space robot system consisting of a three link manipulator.

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* Assistant Professor, Mechanical Engineering Department.
† Lieutenant, United States Navy.

In the recent past, there has been a significant amount of research on the reorientation of planar kinematic chains of rigid multibody systems. Noteworthy among them are Refs. 6–9. In this paper, we consider the reorientation of a space structure in three dimensions as different from planar reorientation. We show that it is possible to use the internal motion of the manipulator to achieve any arbitrary orientation of the space structure. Furthermore, the task of reorientation can be accomplished amidst constraints such as the joint limits of the manipulator. This paper is organized as follows. In Sec. II we summarize some relevant theoretical results. In Sec. III we briefly discuss the planar reorientation of a space structure using a two link manipulator. We extend this approach for the three-dimensional case in Sec. IV, and in Sec. V, we provide some simulation results.

II. Nonholonomic Systems and Their Properties

Nonholonomic mechanical systems are governed by constraints of motion that are nonintegrable differential expressions and are often of the form

$$\sum_{i=1}^{n} v_i \, dx_i = 0 \tag{1}$$

where, the x represent the generalized coordinates and the v are functions of the x. The nonholonomy of a system can be ascertained from the nonintegrable nature of its constraint. In the three-dimensional case, the necessary and sufficient condition for the integrability of the differential equation

$$v_1 \, dx + v_2 \, dy + v_3 \, dz = 0$$

which has the form similar to Eq. (1), is that[10]

$$v_1\left(\frac{\partial v_2}{\partial z} - \frac{\partial v_3}{\partial y}\right) + v_2\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + v_3\left(\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x}\right) = 0 \tag{2}$$

For a dynamical system described by n generalized coordinates and m nonholonomic constraints of the form as in Eq. (1), if we specify the $(n - m)$ independent variables, it is not possible to uniquely determine the remaining m dependent variables. This is because when the independent variables take one set of values from another, the change in the dependent variables depends on the path taken by the independent variables. Therefore, if the independent variables travel along a closed path, the values of the dependent variables at the beginning and end of the path are different. This property of a nonholonomic system is better understood by the use of the following theorem on line integrals.[11]

Theorem: Let $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$, and let v_1, v_2, and v_3 be continuous functions of x, y, and z in a domain D of space. Then the line integral

$$\int_C (v_1 \, dx + v_2 \, dy + v_3 \, dz) \tag{3}$$

is independent of path if and only if the differential form under the integral sign is exact in D, or equivalently the integral is zero for every simple closed path in D, or equivalently

$$\frac{\partial v_2}{\partial z} = \frac{\partial v_3}{\partial y}, \qquad \frac{\partial v_3}{\partial x} = \frac{\partial v_1}{\partial z}, \qquad \frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial x} \tag{4}$$

everywhere in D.

Comparing Eq. (4) (conditions for exactness) to Eq. (2) (conditions for integrability), or directly from the definition of integrability, we know that exactness implies integrability. (Integrability, however, does not imply exactness because an integrable differential expression could have been made exact only after it was multiplied by some integrating factor.) Therefore, it follows that a nonintegrable expression is not exact.

Consider now a nonholonomic system where one of the dependent variables is p and is constrained by the differential expression $dp = v_1 \, dx + v_2 \, dy + v_3 \, dz$, where x, y, and z are the independent variables and v_1, v_2, and v_3 are continuous functions of x, y, and z. Since the system is nonholonomic or nonintegrable, the differential form $v_1 \, dx + v_2 \, dy + v_3 \, dz$ is not exact. This suggests that it is possible to change the coordinates of the dependent variable p of the nonholonomic system using appropriate closed trajectories of the independent variables.

On the basis of the preceding statement we assume that there exists some closed trajectory C of the independent variables x, y, and z that will produce a change in the dependent variable p by some desired amount Δp. If (x_0, y_0, z_0) is any point on this closed trajectory and if the initial configuration of the system is (x_0, y_0, z_0, p_0), then after the system moves once along C, its configuration will be $(x_0, y_0, z_0, p_0 + \Delta p)$. If the closed curve C had been traversed in the opposite direction, then the final configuration of the system would have been $(x_0, y_0, z_0, p_0 - \Delta p)$. Now consider the initial configuration of the system to be (x', y', z', p_0), such that (x', y', z') does not lie on C. Let P be any path segment connecting the point (x', y', z') and any point (x_0, y_0, z_0) on the closed curve C. Let δp denote the change in the dependent variable p, as x, y, and z move along the path segment P from (x', y', z') to (x_0, y_0, z_0). Then, if the system moves from the initial configuration (x', y', z', p_0) to the closed curve C along path P, then moves once along the closed curve C, and finally retraces the path P backwards, the configuration of the system at the end of the path (refer to Fig. 1) will also be $(x', y', z', p_0 + \Delta p)$. This is true because the surface integral of the area bounded by the closed curve beginning and ending at the point (x', y', z') is equal to the surface integral of the area bounded by the closed curve C. From this discussion it follows that the closed curve C that can bring about the desired change in the dependent variable can lie anywhere in the space defined by the independent generalized coordinates.

III. Reorientation of a Rigid Body in Space Using a Manipulator: Planar Case

In this section we review how we can reorient a free-flying rigid body using a manipulator, for the two-dimensional case. A complete discussion on the same can be found in Refs. 6–9. Consider a planar space manipulator consisting of two links mounted on a rigid body.[9] This system can be described by three generalized coordinates: ϕ representing the orientation of the rigid body, and θ_1 and θ_2 representing the joint angles of the manipulator. The two-degrees-of-freedom (2-DOF) system

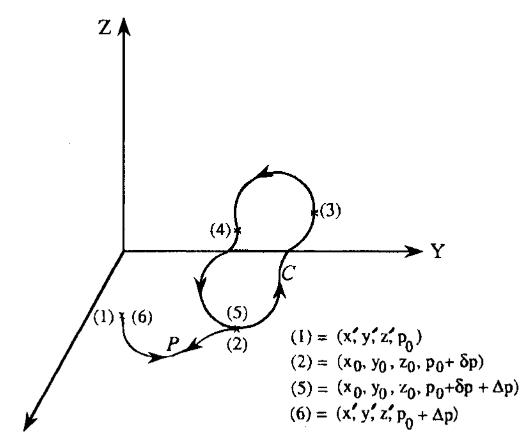

Fig. 1 Closed trajectory C in the independent variables x, y, and z produces a change in the dependent variable p by an amount Δp.

is described by one nonholonomic constraint due to the conservation of angular momentum given by the relation

$$\dot{\phi} = (a\dot{\theta}_1 + b\dot{\theta}_2)/\Delta \tag{5}$$

where Δ, a, and b are functions of the joints angles θ_1 and θ_2 and the system parameters. The expressions for Δ, a, and b can be found in Ref. 9.

The motion planning problem is to find suitable closed trajectories for the joint variables of the manipulator that will change the orientation of the rigid body from its initial value ϕ_i to the desired value ϕ_f. We are interested in closed trajectories of the joint variables of the manipulator because the idea behind our approach is to reorient the rigid body without any other change in the system configuration. For motion of the joints along closed paths, the change in ϕ is represented as

$$\int \mathrm{d}\phi = \oint_C \frac{1}{\Delta}(a\,\mathrm{d}\theta_1 + b\,\mathrm{d}\theta_2) \tag{6}$$

where C is a closed curve in the θ_1-θ_2 plane that we will suitably choose. Using Green's theorem[11] Eq. (6) is simplified to

$$\begin{aligned}\int \mathrm{d}\phi &= \iint_S \left[\frac{\partial}{\partial\theta_1}\left(\frac{b}{\Delta}\right) - \frac{\partial}{\partial\theta_2}\left(\frac{a}{\Delta}\right)\right]\mathrm{d}\theta_1\, d\theta_2 \\ &= MI_0 \iint_S \frac{\partial}{\partial\theta_2}\left(\frac{1}{A + B\cos\theta_2}\right)\mathrm{d}\theta_1\,\mathrm{d}\theta_2\end{aligned} \tag{7}$$

where A and B are defined in Ref. 9, and S is the surface in the θ_1-θ_2 plane confined within the closed curve C. Let the desired change in ϕ be denoted as $\bar{\phi}$. Then the path planning problem reduces to the proper selection of the area S in the θ_1-θ_2 plane such that the following equality is satisfied:

$$\iint_S \frac{\partial}{\partial\theta_2}\left(\frac{1}{A + B\cos\theta_2}\right)\mathrm{d}\theta_1\,\mathrm{d}\theta_2 = \frac{\bar{\phi}}{MI_0} \triangleq k \tag{8}$$

If we choose a rectangular path in the θ_1-θ_2 plane whose sides are parallel to the θ_1 and θ_2 axes, then the preceding identity reduces to

$$(\theta_{1u} - \theta_{1l})\left[\frac{1}{A + B\cos\theta_{2u}} - \frac{1}{A + B\cos\theta_{2l}}\right] = k \tag{9}$$

where, θ_{1l} and θ_{1u} denote the lower and upper extremities of θ_1 in the rectangular path and θ_{2l} and θ_{2u} denote the same for θ_2.

The reachability of the system can be proven by showing that there exists a surface S such that the equality in Eq. (8) can be satisfied for any arbitrary value of k. We first note that if the identity in Eq. (8) can be satisfied for some value of k by traveling along the boundary of the area S in the positive direction, then the same identity can be satisfied for $-k$ by simply traveling along the boundary in the negative direction. Furthermore, if the identity can be satisfied for some value of k by traveling once along the boundary of S, then the identity can be satisfied for the value nk, $n = 1, 2, \ldots$, by traveling n times along the boundary in the same direction. Clearly, the reachability problem reduces to showing that the identity in Eq. (8) can be satisfied for any value of $k \in [0, \epsilon)$, where ϵ is some positive small number. For a rectangular area S, this has been shown in Ref. 9.

IV. Reorientation of a Space Structure Using a Manipulator: Three-Dimensional Case

In our formulation we assume the robot to have a mechanical structure as shown in Fig. 2a. The reference frames in Fig. 2a are according to the Denavit-Hartenburg[12] convention. According to this convention, the joint angles θ_1, θ_2, and θ_3 are measured about the Z_0, Z_1, and Z_2 axes, respectively. Figure 2b depicts the kinematic structure of the manipulator.

The location of the robot with respect to the structure is chosen such that the base frame of the robot, which is fixed to the space structure, is along the principal axes of the space structure. This assumption is quite general because at any point on the structure it will be possible to find a set of principal axes and the robot can be mounted such that the base frame is along these axes. This will allow us to find motions of the manipulator that can produce single rotations of the space structure about its principal axes. Since X_S-Y_S-Z_S are a set of principal axes located at the center of mass of the structure, our simplest choice for the origin of the base frame is the point on the surface of the space structure on the Z_S axis.

A. Particular Configurations of the Three-Link Manipulator

According to the Denavit-Hartenburg convention,[12] the joint angles θ_1, θ_2, and θ_3 are measured about the Z_0, Z_1, and Z_2 axes, respectively. The *home configuration* of the manipulator is an arbitrary configuration where all of the joint angles are referenced to have a zero value, i.e., $(\theta_1, \theta_2, \theta_3) \equiv (0.0, 0.0, 0.0)$. We denote the configuration of the manipulator in Fig. 2a to be the home configuration. We note that in this configuration the entire manipulator lies in the X_S-Z_S plane.

Our goal is to change the orientation of the space structure from an initial set of Euler angles ϕ_{1i}, ϕ_{2i}, ϕ_{3i} to a desired set of values ϕ_{1f}, ϕ_{2f}, ϕ_{3f} without any other change in the system configuration. In other words, we would like the robot manipulator to have the same joint configuration $(\theta_1, \theta_2, \theta_3)$ when the

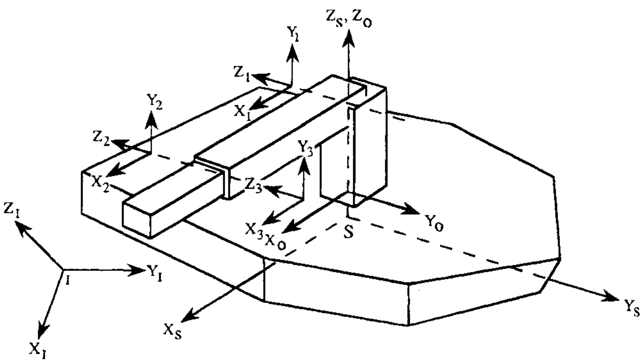

Fig. 2a Home configuration of the three link robot manipulator mounted on the space structure; link frames are according to the Denavit-Hartenburg convention.

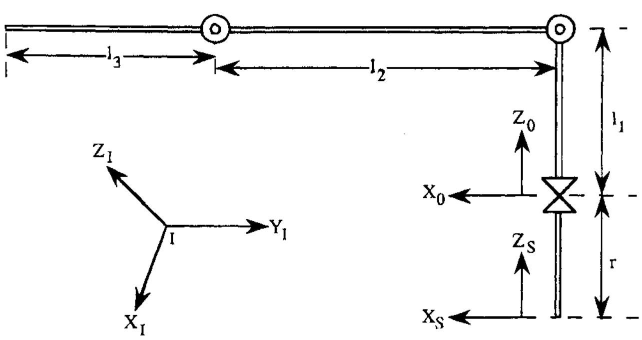

Fig. 2b Kinematic structure of the three link robot manipulator with revolute joints, shown at home configuration.

orientation of the space structure is $(\phi_1, \phi_2, \phi_3) \equiv (\phi_{1i}, \phi_{2i}, \phi_{3i})$, and when $(\phi_1, \phi_2, \phi_3) \equiv (\phi_{1f}, \phi_{2f}, \phi_{3f})$. For convinience, we choose this configuration of the manipulator to be the home configuration.

We define another configuration of the robot manipulator for the simplification of discussion. The *intermediate configuration* of the manipulator is defined by $(\theta_1, \theta_2, \theta_3) \equiv (\pi/2, 0.0, 0.0)$.

When the robot is at its home configuration, if the first joint is kept fixed, the motion of the manipulator is confined to the X_S-Z_S plane of the space structure. By keeping the first joint fixed and moving the second and third joints along closed paths, the robot can return to its home configuration and produce a net rotation of the space structure about its Y_S axis. Starting from the intermediate configuration, if the second and third joints of the robot move along closed paths and the first joint is kept fixed at $\pi/2$, the robot will return to its intermediate configuration although the structure would undergo a net rotation about its X_S axis.

At this point we would like to comment that if arbitrary rotations of the structure about its X_S and Y_S axes can be achieved, then a sequence of rotations about the Y_S, X_S, and Y_S axes can bring about any desired change in the orientation of the space structure.

B. Rotation of the Space Structure About its Y_S Axis: Class $\mathcal{Y}$ Motion

The purpose of the class $\mathcal{Y}$ motion is to change the orientation of the space structure about its Y_S axis using the manipulator. The manipulator will be at the home configuration at the beginning and end of this motion. Furthermore, during this motion the first joint of the manipulator will be kept fixed at $\theta_1 = 0.0$ rad. The motion of the manipulator will remain confined to the X_S-Z_S plane, and the problem will be a planar problem and similar to the one discussed in Sec. III. In this case, the constraint due to the conservation of angular momentum takes the form

$$\dot{\phi} = (a\dot{\theta}_2 + b\dot{\theta}_3)/\Delta \tag{10}$$

$$a \triangleq C_1 \sin\theta_2 + C_2 \cos\theta_3 + C_3 \sin(\theta_2 + \theta_3) + s_1$$

$$b \triangleq C_2 \cos\theta_3 + C_3 \sin(\theta_2 + \theta_3) + s_2$$

$$\Delta \triangleq 2C_1 \sin\theta_2 + C_2 \cos\theta_3 + 2C_3 \sin(\theta_2 + \theta_3) - s_3$$

where the constants C_1, C_2, C_3, s_1, s_2, and s_3 are defined as

$$C_1 \triangleq l_2(0.5\, m_2 + m_3)[m_S(r + l_1) + 0.5\, m_1 l_1]$$

$$C_2 \triangleq m_3 l_2 l_3 (m_S + m_1 + 0.5\, m_2)$$

$$C_3 \triangleq 0.5\, m_3 l_3 [m_S(r + l_1) + 0.5\, m_1 l_1]$$

$$s_1 \triangleq m_t[I_{233} + (0.25\, m_2 + m_3) l_2^2] - (0.5\, m_2 + m_3)^2 l_2^2 + s_2$$

$$s_2 \triangleq m_t[I_{333} + 0.25\, m_3 l_3^2] - 0.25\, m_3^2 l_3^2$$

$$s_3 \triangleq -m_t[I_{S22} + I_{133} + (m_2 + m_3)(r + l_1)^2 + m_1(r + 0.5\, l_1)^2] - [m_1(r + 0.5\, l_1) + (m_2 + m_3)(r + l_1)]^2 - s_1 \tag{11}$$

and where, $m_t \triangleq (m_S + m_1 + m_2 + m_3)$, and r, l_1, l_2, and l_3 are defined in Fig. 2b.

If the second and third joints of the robot move along a closed path C in the θ_2-θ_3 plane, then the net change of orientation of the spacecraft about its Y_S axis will be given by

$$\begin{aligned} \phi_Y &= \int d\phi \\ &= \oint_C \frac{1}{\Delta}(a\, d\theta_2 + b\, d\theta_3) \\ &= \iint_S \left[\frac{\partial}{\partial\theta_2}\left(\frac{b}{\Delta}\right) - \frac{\partial}{\partial\theta_3}\left(\frac{a}{\Delta}\right)\right] d\theta_2\, d\theta_3 \end{aligned} \tag{12}$$

where S is the surface area enclosed within the closed curve C. As in Sec. III, we choose the surface area S to be rectangular to simplify the integration. Furthermore, we specify our rectangular area arbitrarily by the limits $0 \le \theta_2 \le \theta_{2u}$, and $0 \le \theta_3 \le 3\pi/4$. For this choice of S, the first term in the surface integral of Eq. (12) reduces to

$$\begin{aligned} \iint_S \frac{\partial}{\partial\theta_2}\left(\frac{b}{\Delta}\right) d\theta_2\, d\theta_3 &= \int_0^{3\pi/4}\left[\left(\frac{b}{\Delta}\right)_{\theta_2=\theta_{2u}} - \left(\frac{b}{\Delta}\right)_{\theta_2=0}\right] d\theta_3 \\ &= \frac{1}{2}\int_0^{3\pi/4}\left[\frac{(2s_2 + s_3 - 2C_1\sin\theta_{2u}) + C_2\cos\theta_3}{(2C_1\sin\theta_{2u} - s_3) + (C_2 + 2C_3\sin\theta_{2u})\cos\theta_3 + 2C_3\cos\theta_{2u}\sin\theta_3}\right] d\theta_3 \\ &\quad - \frac{1}{2}\int_0^{3\pi/4}\left[\frac{(2s_2 + s_3) + C_2\cos\theta_3}{-s_3 + C_2\cos\theta_3 + 2C_3\sin\theta_3}\right] d\theta_3 \end{aligned} \tag{13}$$

and the second term reduces to

$$\begin{aligned} \iint_S -\frac{\partial}{\partial\theta_3}\left(\frac{a}{\Delta}\right) d\theta_2\, d\theta_3 &= \int_0^{\theta_{2u}}\left[\left(\frac{a}{\Delta}\right)_{\theta_3=0} - \left(\frac{a}{\Delta}\right)_{\theta_3=3\pi/4}\right] d\theta_2 \\ &= \frac{1}{2}\int_0^{\theta_{2u}}\left[\frac{2s_1 + s_3 + C_2}{(C_2 - s_3) + 2(C_1 + C_3)\sin\theta_2}\right] d\theta_2 \\ &\quad - \frac{1}{2}\int_0^{\theta_{2u}}\left[\frac{\sqrt{2}(2s_1 + s_3) - C_2}{-(\sqrt{2}s_3 + C_2) + 2C_3\cos\theta_2 + 2(\sqrt{2}C_1 - C_3)\sin\theta_2}\right] d\theta_2 \end{aligned} \tag{14}$$

Equations (12) and (14) can be integrated using integration tables in Ref. 13. The end result is obtained with ϕ_Y being expressed as a continuous function of θ_{2u}. For the sake of brevity we provide the relation showing the functional dependence of ϕ_Y on θ_{2u} in Eq. (A2) of the Appendix. The variation of ϕ_Y with θ_{2u}, $0 \le \theta_{2u} < \pi$, is shown in Fig. 3. Since the joints of a robot manipulator will have physical limits, the maximum absolute value of ϕ_Y will be limited by the maximum value of θ_{2u}, as evident from Fig. 3. For example, if we impose the limit

Table 1 Dynamic parameters of the three link manipulator

		$k = S$	$k = 1$	$k = 2$	$k = 3$
I_{kij}, kg-m^2	(1,1)	23.95781	00.0830	00.0147	00.0117
	(2,2)	13.87031	00.0103	00.2343	00.1221
	(3,3)	37.82812	00.0830	00.2343	00.1221
m_k, kg		302.6250	7.62615	10.8945	8.71560

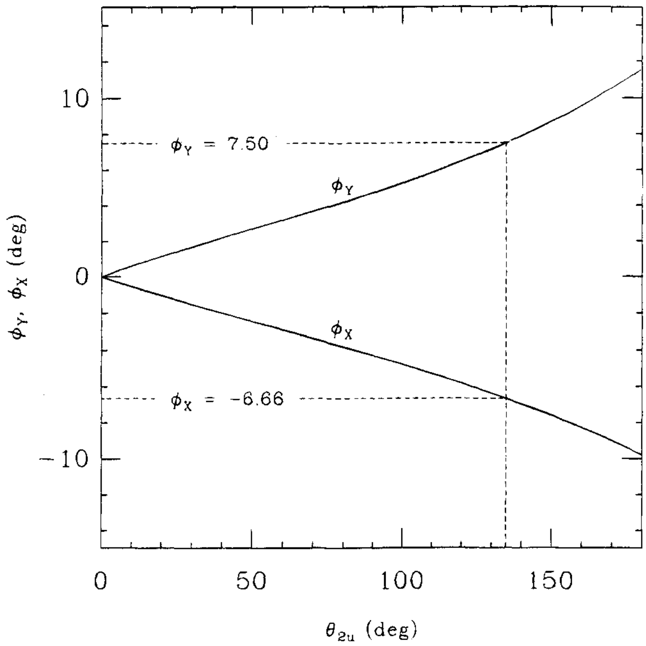

Fig. 3 For the simulation discussed in Sect.V, the change in the orientation of the space structure about its x and y axes: ϕ_X and ϕ_Y, respectively, depend upon the dimension of the closed rectangular path in the θ_2-θ_3 plane.

$\theta_{2u} \leq 3\pi/4$ rad, we observe from Fig. 3 that the maximum change in the orientation of the space structure will be of the order of +7.50 deg. The sign is not our concern here because the sign can be reversed by simply traversing the closed path in the θ_2-θ_3 plane in the opposite direction. If a change in the orientation of magnitude greater than 7.50 deg is desired, the manipulator joints will have to move along some closed path a multiple number of times. Then, the choice for θ_{2u} for this closed path can be made in the following way.

Let Ω be the absolute value of the desired change in orientation of the structure in the Y_S direction. We assume that the second joint of the robot has a physical joint limit of $3\pi/4$ rad, and let us try to minimize n, the number of times the robot manipulator has to move along the closed path. The integer n is then obtained by maximizing δ where

$$\delta = (\Omega/n) \leq 7.50$$

The value of θ_{2u} is subsequently obtained with the help of Fig. 3. The value of θ_{2u} corresponds to the value of $\phi_Y = \delta$. We use a nonlinear function solver to solve for the root of Eq. (A2), with $\phi_Y = \delta$.

The preceding discussion convinces us that any changes of orientation of the space structure about its Y_S axis can be achieved through single or multiple closed-loop trajectories of class $\mathcal{Y}$ motion.

It may be noted that the plot of ϕ_Y vs θ_{2u} depends on the parameters of the robot. The plot shown in Fig. 3 was obtained for the kinematic and dynamic parameters given later in Eq. (22) and in Table 1.

C. Rotation of the Space Structure About its Z_S Axis: Class $\mathcal{Z}$ Motion

We mentioned at the end of Sec. IV.B that the space structure can undergo an arbitrary change in its orientation if it goes through a sequence of rotations about the Y_S, X_S, and Y_S axes. After a class $\mathcal{Y}$ motion, during which the structure will rotate about its Y_S axis, the manipulator will return to its home configuration. It will then have to be reconfigured to its intermediate configuration before it can carry out a rotation of the structure about its X_S axis. Similarly, after the completion of rotation about the X_S axis the manipulator will have to be reconfigured from its intermediate configuration back to the home configuration for the final rotation of the structure about its Y_S axis.

The purpose of the class $\mathcal{Z}$ motion is only to reconfigure the manipulator. This motion will be used to bring the manipulator to the intermediate configuration from the home configuration, and vice versa. In this process the space structure will undergo some rotation about its Z_S axis. The reconfiguration will be achieved by using only the first joint of the manipulator. The second and third joints of the manipulator will be held fixed at $(\theta_2, \theta_3) \equiv (0, 0)$ during this motion.

When the manipulator is at the home configuration or at the intermediate configuration, and moves only its first joint, the angular momentum constraint takes the holonomic form

$$I_{S33}\dot{\phi}_Z + [I_{122} + I_{222} + I_{322} + 0.25m_2l_2^2 + m_3(0.5l_3 + l_2)^2] (\dot{\theta}_1 + \dot{\phi}_Z) = 0$$

which on integration results in

$$\phi_Z = \pm\left(\frac{I_M}{I_{S33} + I_M}\right)\frac{\pi}{2} \tag{15}$$

$$I_M \triangleq I_{122} + I_{222} + I_{322} + 0.25m_2l_2^2 + m_3(0.5l_3 + l_2)^2$$

The change in the orientation ϕ_Z in Eq. (15) is positive when the robot moves from the intermediate configuration to the home configuration. It is negative when the robot moves from the home configuration to the intermediate configuration. This can be ascertained from Fig. 2a, where Z_0 shows the direction in which θ_1 is measured positive. The absolute value of ϕ_Z is a constant whose values depend on the inertia parameters of the system. In Eq. (15), the terms I_{S33}, I_{122}, I_{222}, and I_{322} are in accordance with the definition of the Denavit-Hartenburg link frames in Fig. 2a. These terms were defined in the Nomenclature.

D. Rotation of the Space Structure about Its X_S Axis: Class $\mathcal{X}$ Motion

The purpose of the class $\mathcal{X}$ motion is to change the orientation of the space structure about its X_S axis using the manipulator. The manipulator will be at the intermediate configuration at the beginning and end of this motion. Furthermore, during this motion the first joint of the manipulator will be kept fixed at $\theta_1 = \pi/2$ rad. The motion of the manipulator will therefore remain confined to the Y_S-Z_S plane. The problem will be very similar to the one discussed in Sec. IV.C, and in this case the second and third joints of the manipulator will bring about a change in the orientation of the space structure about its X_S axis. All of the equations developed in Sec. IV.C will apply. The only changes that need to be made are (1) the constant s_3 in Eq. (11) has to be redefined with I_{S22} replaced by I_{S11}, and (2) Δ in Eq. (10) has to be replaced by $-\Delta$. I_{S22} will have to be replaced by I_{S11} because the structure will now rotate about the X_S axis instead of the Y_S axis. The reason for replacing Δ by $-\Delta$ can be explained as follows. During class $\mathcal{Y}$ motion when the manipulator is confined to the X_S-Z_S plane, a positive rotation of the second or third joint will cause the structure to rotate in the direction of positive Y_S axis, but during class $\mathcal{X}$ motion when the manipulator will be confined to the Y_S-Z_S plane, a positive rotation of the second or third joint will cause the structure to rotate in the direction of negative X_S axis.

Using Eqs. (10–14), the net change in the orientation of the space structure about its X_S axis, to be denoted by ϕ_X, can be expressed as a continuous function of θ_{2u}. The variation of ϕ_X with θ_{2u}, $0 \leq \theta_{2u} < \pi$, is shown in Fig. 3 for the set of parameters given later in Eq. (22) and Table 1. If we impose the limit of $\theta_{2u} \leq 3\pi/4$ rad, then we find from Fig. 3 that the maximum change in the orientation of the structure about its X_S axis is limited to 6.66 deg.

Using the same logic that we have used in Sec. IV.C, we can conclude that any arbitrary change in the orientation of the space structure about its X_S axis can be achieved.

E. Synthesis of Manipulator Motion for Reorientation of the Space Structure

As mentioned in Sec. IV.B, the manipulator will be at its home configuration at the initial point of time. After a sequence of motion of its joints the robot will return to the home configuration. This sequence of motion is expected to reorient the spacecraft. Let the initial orientation and the desired orientation of the spacecraft with respect to the inertia frame be given by the rotation matrices $\boldsymbol{R}_i$ and $\boldsymbol{R}_f$, respectively. Then,

$$\boldsymbol{R}_i \triangleq \boldsymbol{R}[y, \phi_{3i}]\boldsymbol{R}[x, \phi_{2i}]\boldsymbol{R}[y, \phi_{1i}]$$

$$\boldsymbol{R}_f \triangleq \boldsymbol{R}[y, \phi_{3f}]\boldsymbol{R}[x, \phi_{2f}]\boldsymbol{R}[y, \phi_{1f}]$$

where $(\phi_{1i}, \phi_{2i}, \phi_{3i})$, and $(\phi_{1f}, \phi_{2f}, \phi_{3f})$ denote the set of *y-x-y* Euler angles describing the initial and the desired orientation of the space structure with respect to the inertia frame. Then, the set of *y-x-y* Euler angles (ϕ_1, ϕ_2, ϕ_3) describing the desired orientation of the space structure with respect to the initial orientation can be solved from the following equation:

$$\boldsymbol{R}[y, \phi_3]\boldsymbol{R}[x, \phi_2]\boldsymbol{R}[y, \phi_1] = \boldsymbol{R}_f\boldsymbol{R}_i^T \tag{16}$$

Equation (16) has a singularity for $\phi_2 = 0, \pm\pi$. Except for this situation, ϕ_1, ϕ_2, and ϕ_3 can be solved uniquely from Eq. (16). At the singular configuration(s), the orientation of the structure can be trivially depicted by one single rotation about the Y_S axis of magnitude $(\phi_1 + \phi_3)$ for $\phi_2 = 0$ and of magnitude $(\phi_1 - \phi_3)$ for $\phi_2 = \pm\pi$.

Now consider the following motion sequence of the manipulator.

1) Class $\mathcal{Y}$ motion with $\phi_Y = \Lambda_1$. The change in the orientation of the space structure can be represented by $\boldsymbol{R}[y, \Lambda_1]$. At the end of this motion the manipulator returns to the home configuration.

2) Class $\mathfrak{Z}$ motion with $\phi_Z = \Lambda_2$. Λ_2 is obtained from Eq. (15) as

$$\Lambda_2 = -\left(\frac{I_M}{I_M + I_{S33}}\right)\frac{\pi}{2} \tag{17}$$

The change in the orientation of the space structure can be represented by $\boldsymbol{R}[z, \Lambda_2]$. By virtue of this motion, the manipulator moves from the home configuration to the intermediate configuration.

3) Class $\mathcal{X}$ motion with $\phi_X = \Lambda_3$. The change in the orientation of the space structure can be represented by $\boldsymbol{R}[x, \Lambda_3]$. At the end of this motion the manipulator returns to the intermediate configuration.

4) Class $\mathfrak{Z}$ motion with $\phi_Z = -\Lambda_2$, where Λ_2 is defined by Eq. (17). The change in the orientation of the space structure can be represented by $\boldsymbol{R}[z, -\Lambda_2]$. By virtue of this motion, the manipulator moves from the intermediate configuration to the home configuration.

5) Class $\mathcal{Y}$ motion with $\phi_Y = \Lambda_4$. The change in the orientation of the space structure can be represented by $\boldsymbol{R}[y, \Lambda_4]$. At the end of this motion the manipulator returns to the home configuration.

If the manipulator goes through the sequence of motions discussed, the change in the orientation of the space structure would be represented by the rotation matrix $\boldsymbol{R}[y, \Lambda_4]\boldsymbol{R}[z, -\Lambda_2]$ $\boldsymbol{R}[x, \Lambda_3]\boldsymbol{R}[z, \Lambda_2]\boldsymbol{R}[y, \Lambda_1]$. If an arbitrary change in the orientation of the space structure given by Eq. (16) is to be attained through the sequence of motions given, then we should be able to solve for Λ_1, Λ_3, and Λ_4 from the following equation:

$$\begin{aligned}&\boldsymbol{R}[y, \Lambda_4]\boldsymbol{R}[z, -\Lambda_2]\boldsymbol{R}[x, \Lambda_3]\boldsymbol{R}[z, \Lambda_2]\boldsymbol{R}[y, \Lambda_1]\\ &\quad= \boldsymbol{R}[y, \phi_3]\boldsymbol{R}[x, \phi_2]\boldsymbol{R}[y, \phi_1]\end{aligned} \tag{18}$$

for arbitrary values of ϕ_1, ϕ_2, and ϕ_3. Equation (18) has a singularity for $\phi_2 = 0, \pm\pi$. Then Eq. (18) can be solved by setting $\Lambda_2 = \Lambda_3 = \Lambda_4 = 0$, and equating $\Lambda_1 = (\phi_1 + \phi_3)$ when $\phi_2 = 0$, and $\Lambda_1 = (\phi_1 - \phi_3)$ when $\phi_2 = \pm\pi$. When $\phi_2 \neq 0, \pm\pi$, we solve for Λ_1, Λ_3, and Λ_4 by first rewriting Eq. (18) as

$$\begin{aligned}&\boldsymbol{R}[z, -\Lambda_2]\boldsymbol{R}[x, \Lambda_3]\boldsymbol{R}[z, \Lambda_2]\\ &\quad= \boldsymbol{R}[y, \phi_3 - \Lambda_4]\boldsymbol{R}[x, \phi_2]\boldsymbol{R}[y, \phi_1 - \Lambda_1], \quad \phi_2 \neq 0, \pm\pi\end{aligned} \tag{19}$$

The product of the matrices on both sides of Eq. (19) is a direction cosine matrix that can be equivalently represented by the set of four Euler parameters[14] β_0, β_1, β_2, and β_3 as follows:

$$\begin{aligned}
\beta_0 &= \cos\left(\frac{\Lambda_3}{2}\right) & \beta_0 &= \cos\left(\frac{\phi_2}{2}\right)\cos\left(\frac{\phi_1 - \Lambda_1 + \phi_3 - \Lambda_4}{2}\right)\\
\beta_1 &= \sin\left(\frac{\Lambda_3}{2}\right)\cos\Lambda_2 & \beta_1 &= \sin\left(\frac{\phi_2}{2}\right)\cos\left(\frac{\phi_3 - \Lambda_4 - \phi_1 + \Lambda_1}{2}\right)\\
\beta_2 &= \sin\left(\frac{\Lambda_3}{2}\right)\sin\Lambda_2 & \beta_2 &= \cos\left(\frac{\phi_2}{2}\right)\sin\left(\frac{\phi_1 - \Lambda_1 + \phi_3 - \Lambda_4}{2}\right)\\
\beta_3 &= 0 & \beta_3 &= \sin\left(\frac{\phi_2}{2}\right)\sin\left(\frac{\phi_3 - \Lambda_4 - \phi_1 + \Lambda_1}{2}\right)
\end{aligned}$$

Since $\phi_2 \neq 0, \pm\pi$, Eq. (20) can be solved for Λ_1, Λ_3, and Λ_4 as follows:

$$\begin{aligned}\Lambda_3 &= 2\arcsin[\sin(\phi_2/2)\sec\Lambda_2]\\ \Lambda_1 &= \phi_1 - \arctan[\sin\Lambda_2\tan(\Lambda_3/2)]\\ \Lambda_4 &= \Lambda_1 + \phi_3 - \phi_1\end{aligned} \tag{21}$$

The algorithm for the reorientation of the spacecraft, therefore, can be established as follows. First solve for the necessary change in the orientation ϕ_1, ϕ_2, and ϕ_3 from Eq. (16). Next, using the computed values of ϕ_1, ϕ_2, and ϕ_3, compute the values of Λ_1, Λ_3, and Λ_4 from Eq. (21). For each of Λ_1, Λ_3, and Λ_4, compute the closed trajectory in the ϕ_2-ϕ_3 plane and the number of times that the manipulator has to traverse the closed trajectory. Such trajectories can always be planned, as we have seen in Secs. IV.C and IV.E. Finally, follow the five step motion sequence.

V. Simulation

We assume the manipulator in Fig. 2b to have the following kinematic parameters:

$$r = 0.15\text{ m},\quad l_1 = 0.35\text{ m},\quad l_2 = 0.5\text{ m},\quad l_3 = 0.4\text{ m} \tag{22}$$

For the same manipulator, whose Denavit-Hartenburg reference frames are shown in Fig. 2a, we used the values listed in Table 1 for its dynamic parameters.

The initial and the desired orientations of the space structure were

$$(\phi_{12}, \phi_{2i}, \phi_{3i}) \equiv (135.0, 25.0, -105.0)$$

$$(\phi_{1f}, \phi_{2f}, \phi_{3f}) \equiv (-55.0, 95.0, 75.0)$$

where the units are in degrees. We solve for the *y-x-y* Euler angles from Eq. (16) as

$$(\phi_1, \phi_2, \phi_3) \equiv (-86.47308, 119.57773, 70.15945)$$

Then, Λ_1, Λ_3, and Λ_4 can be obtained from Eq. (21), and Λ_2 from Eq. (17)

$$\Lambda_1 = -66.79483, \qquad \Lambda_2 = -11.09346$$

$$\Lambda_3 = 123.43739, \qquad \Lambda_4 = 89.83769$$

where the units are in degrees.

1) Class $\mathcal{Y}$ motion with $\Lambda_1 = -66.79483$ deg. Assuming a joint limit of $3\pi/4$ rad on the second joint, the minimum number of times the robot has to move along a closed trajectory will be $n = 9$. Then, for each closed-loop motion the change in the orientation needs to be $-66.79483/9 = -7.42164$ deg. From Fig. 3, we find that $\phi_Y = +7.42164$ deg corresponds to a value of θ_{2u} that lies between 125.0 and 135.0 deg. Using these values as the lower and upper limits, we find the exact solution for $\phi_Y = 7.42164$ in Eq. (A2) to be 133.84235 deg. The negative sign in the change in orientation can be taken care of by simply travelling along the closed path in the negative direction. In Fig. 4, ABCDA denotes the directed closed-loop path in the θ_2-θ_3 plane. The change in the *y-x-y* Euler angles (ϕ_1, ϕ_2, ϕ_3) is shown in Fig. 5 during the time from $t = 0$ s to $t = 483.92$ s. It can be seen from Fig. 5 that during this time ϕ_1 and ϕ_2 remain constant whereas ϕ_3 changes with a periodic motion. The number of periods are equal to nine, and this corresponds to the number of times the second and third joints of the robot move along the closed path ABCDA in Fig. 4.

2) Class $\mathcal{Z}$ motion with $\Lambda_2 = -11.09346$ deg. In Fig. 4, the path segment AO corresponds to this motion. The variation of the *y-x-y* Euler angles during this motion are not very clear from Fig. 5 because this motion takes only 10 s to complete, as compared to the total time of simulation which is of the order of 2166 s.

3) Class $\mathcal{X}$ motion with $\Lambda_3 = 123.43739$ deg. Assuming a joint limit of $3\pi/4$ rad on the second joint, the minimum number of times the robot has to move along a closed trajectory will be $n = 19$. Then, for each closed-loop motion the change in the orientation needs to be $123.43739/19 = 6.49670$ deg. From Fig. 3, we find that $\phi_X = -6.49670$ deg corresponds to a value of θ_{2u} that lies between 125.0 and 135.0 deg. Using these values as the lower and upper limits, we find the exact solution for $\phi_X = -6.49670$ to be 132.19918 deg. Since traveling along the positive direction of the closed path produces a negative change in the orientation ϕ_X as evident from Fig. 3, we will travel in the negative direction. In Fig. 4, OPQRO denotes the directed closed-loop path in the θ_2-θ_3 plane. The change in the *y-x-y* Euler angles (ϕ_1, ϕ_2, ϕ_3) is shown in Fig. 5 during the time from $t = 493.92$ s to $t = 1509.27$ s. It can be seen from the figure that all of the Euler angles undergo a periodic motion during this time. The number of periods can be seen to be equal to 19 and equals the number of times the second and third joints of the robot move along the closed path OPQRO in Fig. 4.

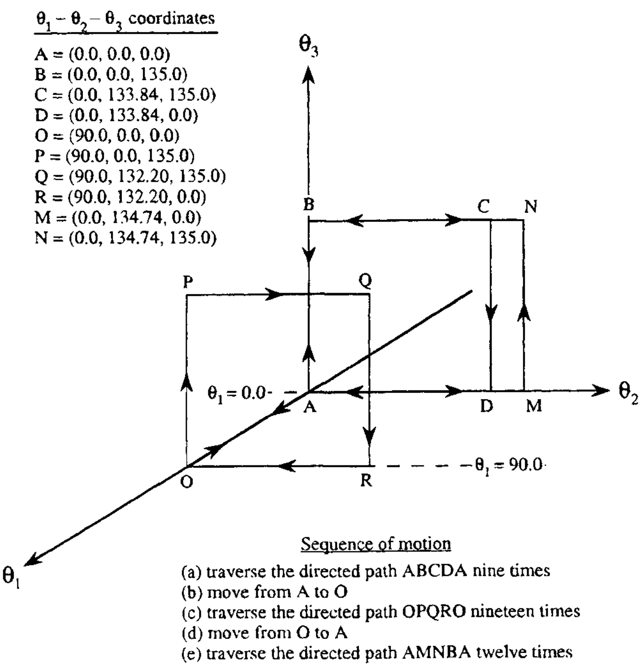

Fig. 4 Description of the closed-loop path in the θ_1-θ_2-θ_3 space that changes the orientation of the space structure from an initial *y-x-y* Euler angles of (135.0, 25.0, − 105.0) deg to a final value of (−55.0, 95.0,75.0) deg.

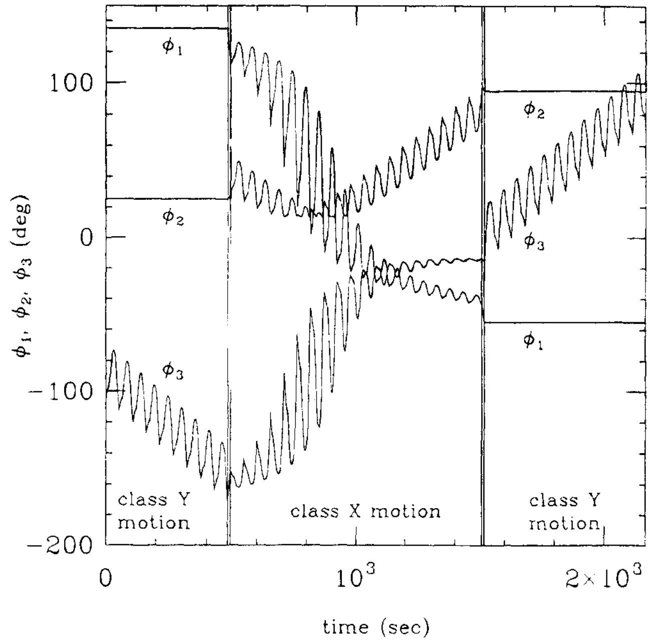

Fig. 5 Evolution of the Euler angles ϕ_1, ϕ_2, ϕ_3 describing the orientation of the space structure, for the simulation discussed in Sec. V.

4) Class $\mathcal{Z}$ motion with $\Lambda_2 = 11.09346$ deg. In Fig. 4, the path segment OA corresponds to this motion. The variation of the *y-x-y* Euler angles during this motion are not very clear from Fig. 5 because this motion takes only 10 s to complete, as compared to the total time of simulation which is of the order of 2166 s.

5) Class $\mathcal{Y}$ motion wih $\Lambda_1 = 89.83769$ deg. Assuming a joint limit of $3\pi/4$ rad on the second joint, the minimum number of times the robot has to move along a closed trajectory will be $n = 12$. Then, for each closed-loop motion the change in the orientation needs to be $89.83769/12 = 7.48647$ deg. From Fig. 3, we find that $\phi_Y = 7.48674$ deg corresponds to a value of θ_{2u} that lies between 125.0 and 135.0 deg. Using these values as the lower and upper limits, we find the exact solution for $\phi_Y = 7.48674$ in Eq. (A2) to be 134.73799 deg. In Fig. 4, AMNBA denotes the directed closed-loop path in the θ_2-θ_3 plane. The change in the *y-x-y* Euler angles (ϕ_1, ϕ_2, ϕ_3) is shown in Fig. 5 during the time from $t = 1519.27$ s to $t = 2166.64$ s. It can be seen from the figure that during this time the Euler angles ϕ_1 and ϕ_2 remain constant whereas ϕ_3 changes with a periodic motion. The number of periods can be seen to be equal to 12 and it equals the number of times the second and third joints of the robot move along the closed path AMNBA in Fig. 4.

VI. Concluding Remarks

We presented in this paper a new method for the three-dimensional reorientation of space structures. In our method we used a three link serial manipulator for the reorientation of the space structure. It was shown that the nonholonomic mechanical structure of the system comprising of the space structure and the manipulator can be utilized to achieve this task. We adopted a surface integral approach for the motion planning of the manipulator where the physical joint limits of the manipulator were taken into consideration. The results of simulation of the reorientation of a space structure were provided.

It may be noted that our motion planning algorithm provide us with only the joint trajectories of the manipulator and, therefore, the joint velocities and accelerations can be chosen arbitrarily. Since the time history of the joint trajectories are unspecified, we can choose the time scale suitable to our needs. For a rigid body system, larger magnitudes of joint velocities and accelerations can be chosen depending on the actuator capabilities. This will reduce the total time taken for the attitude maneuver. In our simulations we selected the joint trajectories to have a uniform velocity of 10 deg/s and the total time for the simulation was approximately 36 min.

Appendix: Rotation of the Space Structure about Its Y Axis: ϕ_Y

In addition to Eq. (11), where the constants C_1, C_2, C_3, s_1, s_2, and s_3 are defined, we define the following

$$
\begin{aligned}
v_1 &\triangleq C_2^2 + 2C_3^2 \\
v_2 &\triangleq C_2^2 + 4C_3^2 \\
v_3^2 &\triangleq s_3^2 - C_2^2 - 4C_3^2 \\
v_4 &\triangleq v_1 + 3C_2C_3 \sin\theta_{2u} \\
v_5 &\triangleq v_2 + 2C_2C_3 \sin\theta_{2u} \\
v_6^2 &\triangleq (s_3 - 2C_1\sin\theta_{2u})^2 - (C_2 + 2C_3\sin\theta_{2u})^2 - 4C_3^2\cos^2\theta_{2u} \\
v_7^2 &\triangleq s_3^2 - 4C_3^2 - 4C_1^2 \\
v_8^2 &\triangleq (C_2 - s_3)^2 - 4(C_1 + C_3)^2
\end{aligned}
\tag{A1}
$$

Since the quantities v_3^2, v_6^2, v_7^2, and v_8^2 are always positive, ϕ_Y in Eq. (12) can be expressed as

$$
\begin{aligned}
\phi_Y = &-\frac{C_2C_3\cos\theta_{2u}}{v_5}\, ln\left(\frac{s_3 - 2C_1\sin\theta_{2u} - 2C_3\cos\theta_{2u}}{s_3 - C_2 - 2(C_1 + C_3)\sin\theta_{2u}}\right) - \frac{\pi v_4}{2v_5} \\
&+ \frac{2[s_2v_5 + (s_3 - 2C_1\sin\theta_{2u})v_4]}{v_5v_6}\arctan\left(\frac{s_3 + C_2 - 2(C_1 - C_3)\sin\theta_{2u} - 2C_3\cos\theta_{2u}}{v_6}\right) \\
&- \frac{2[s_2v_5 + (s_3 - 2C_1\sin\theta_{2u})v_4]}{v_5v_6}\arctan\left(\frac{-2C_3\cos\theta_{2u}}{v_6}\right) + \frac{v_1}{v_2}\left[\frac{\pi}{2} + ln\left(\frac{2C_3 - s_3}{C_2 - s_3}\right)\right] \\
&+ \frac{2(s_2v_2 + s_3v_1)}{v_2v_3}\left[\arctan\left(\frac{2C_3 - C_2 - s_3}{v_3}\right) - \arctan\left(\frac{2C_3}{v_3}\right)\right] \\
&+ \frac{s_3 + 2s_1}{v_7}\left[\arctan\left(\frac{2C_1 - (s_3 + 2C_3)\tan(\theta_{2u}/2)}{v_7}\right) - \arctan\left(\frac{2C_1}{v_7}\right)\right] \\
&- \frac{s_3 + 2s_1 + C_2}{v_8}\left[\arctan\left(\frac{2(C_1 + C_3) + (C_2 - s_3)\tan(\theta_{2u}/2)}{v_8}\right) - \arctan\left(\frac{2(C_1 + C_3)}{v_8}\right)\right]
\end{aligned}
\tag{A2}
$$

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